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N-PERSON GAME THEORY

L. S. Shapley

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L. S. Shapley

> ...in a world with multiple interests, sometimes in conflict and sometimes in cooperation with one another, all the facts and quantifications by themselves do not necessarily point unambiguously to a "correct" decision, or to a "fair" allocation of resources. Indeed, the fundamental meaning of the words "correct" and "fair" is a matter of judgment in a multipolar world. That this is the case is made particularly clear by the theory of n-person games. ...[The next speaker] will discuss some of the basic concepts of n-person game theory, with illustrations from theoretical economics.
>
> (from the moderator's preface)

In this brief talk I can only hope to give a taste of the current trends of research into the theory of n-person games, not a well-rounded survey. I shall try to define two or three of the key concepts in the theory, and then apply them to some absurdly simple mathematical models of multilateral competitive situations. I would emphasize, however, that my object will be to illustrate the theory itself, not to present applications.

First, some general remarks about the term "n-person:" A <u>zero</u>-person game, if you will, is a mechanical model, or a behavioristic model if human agents are involved. A <u>one</u>-person game is the standard decision problem, with perhaps "Nature" (a non-player) personifying the element of uncertainty faced

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by the decisionmaker. With two or more players, a quite different form of indeterminacy appears, due to the exercising of free choice by independent agents. With three or more players, coalition-forming becomes a significant, sometimes decisive, possibility, and it is at this point that we enter the characteristic domain of "n-person game theory" (Chart 1).

Typically, the various interests in an n-person game are at cross-purposes. Parallel interests (as in the theory of teams) or directly opposed interests (as in the theory of zero-sum, two-person games) tend to wipe out the coalitional question, and hence to permit the more explicit methods of direct optimization and minimax to be brought to bear--tools that are not of much help against the difficulties of the general "n-person problem."

N-person theory, then, is concerned with things like cooperation, coalition, organizational structure, commitment, trust, compromise, threat, enforceability, and indeed the whole legal/social/cultural environment. It de-emphasizes questions of tactical optimization, the detailed spelling out of rules, and the numerical calculation of outcomes and payoffs. Nevertheless, the theory remains heavily mathematical.

The cornerstone of the theory is the concept of the characteristic function of a game--a fundamental idea due to the mathematician John von Neumann (Chart 2). The characteristic function puts a umerical value on each potential coalition of players. It is remarkable not for what it says about the game from which it is calculated, but for what it does not say. In reducing a game to its characteristic function, virtually all of the details of moves, information, timing, and payoffs are suppressed. Yet, as von Neumann saw, the heart of the "n-person problem" remains, stripped of all strategic distractions.

The characteristic function is not adequate for all games, however. There are two important conditions for its use: (1) Money must be present, or something that acts like money. Otherwise the potential of a coalition could not possibly be boiled down to a single number, representing a utility freely

- COLD WAR
- INTERNATIONAL TRADE
- ELECTORAL POLITICS
- MARKETS
- CORPORATE OWNERSHIP
- BUREAUCRACIES
- NATIONAL GOALS

Chart 1—Areas for multilateral-decision models

S = ANY POTENTIAL COALITION

v(S) = HOW MUCH S CAN WIN,
INDEPENDENTLY OF
OTHER PLAYERS

MAJOR ASSUMPTIONS

1 "MONEY"

2 NO COSTLY THREATS

Chart 2—The characteristic function "v"

sharable among its members. (2) Threats costly to carry out must not be a determining factor in the coalitional interplay. The characteristic function pessimistically assumes that each coalition will experience the most damaging countermoves by the rest of the players; yet costly threats are always negotiable.

Generalizations and extensions of the characteristic function idea have been developed, circumventing these restrictions; I cannot hope to go into them here. Fortunately, interesting classes of characteristic-function games exist, with money and without threats, particularly in the field of economics. Indeed, the use of money is almost a definition of economic activity, while the "worst threat" in many economic situations is merely to disengage--to take one's trade elsewhere or go into business for oneself.

The characteristic function is of course just a beginning--a descriptive tool, a classifying tool, a "pre-solution." In the past two decades, many notions of "solution" of a game have been devised, mostly based on the characteristic function, and half-a-dozen continue to receive serious attention from mathematicians, economists, and political scientists. A pluralistic theory has arisen; each solution concept, in its own way, addresses some aspect of the "n-person problem" (Chart 3).

The Pareto set, a familiar concept from economic theory, merely identifies the outcomes that are not subject to simultaneous improvement for all parties.

The core extends this principle to all subsets of players: no coalition, through its own efforts, can improve the lot of all its members, if the status quo is "in the core." But many games do not have cores, and thus are inherently unstable. As a general rule, games of economic origin do have cores, while games based on political models do not.

The value concerns not stability, but bargaining position. It provides an a priori assessment of the utility in becoming involved in a game. I shall have more to say about the value presently.

In political games, where control rather than money is the payoff, the value has been reinterpreted as a power index, and has been widely used as a

- PARETO SET — Outcomes that can't be improved upon for all players
- CORE — Outcomes that satisfy every coalition
- VALUE — A priori worth of each player's "seat" (average amount he adds to a coalition)
- POWER INDEX — Probability of pivotal vote
- STABLE SETS
- BARGAINING SETS

} — Self-consistent standards of behavior

. . . ETC.

<u>ALSO</u>

- COMPETITIVE EQUILIBRIUM

Chart 3—Several solution concepts

test of the "fairness" of electoral and legislative systems.

Other popular solution concepts, such as the bargaining sets and the stable sets of von Neumann and Morgenstern, seek to identify distributions or configurations that are stable or "rational" under various criteria. Also, the competitive equilibrium of classical economics, though a behavioristic construct, nevertheless often makes an interesting comparison with one or more of the game-theoretic solutions, in models where both can be defined.

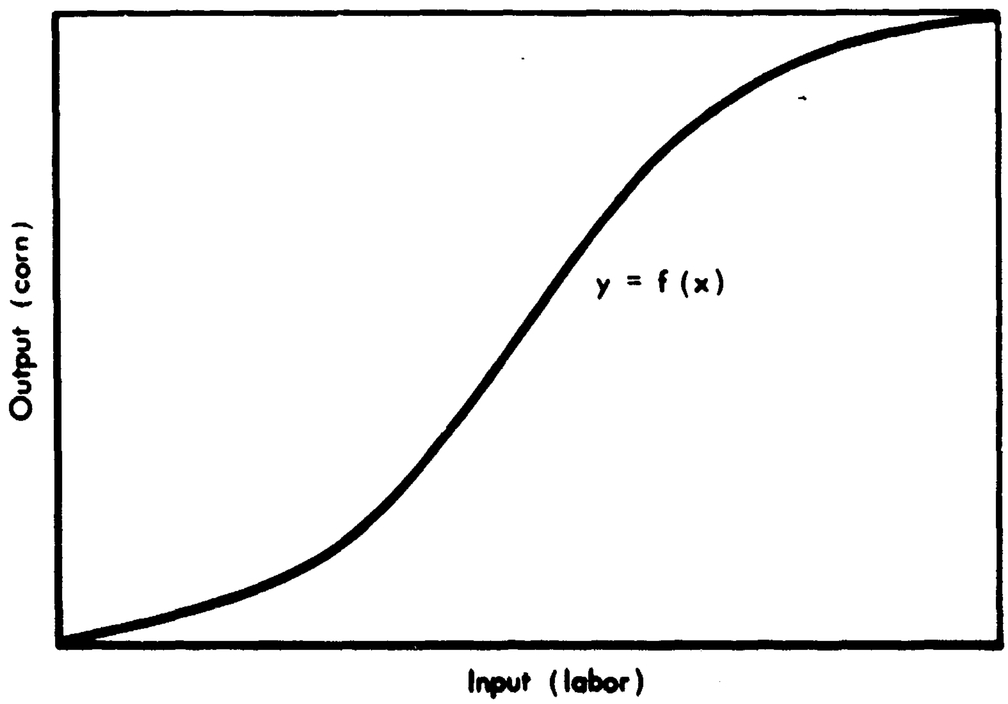


Chart 4—A production function

I have chosen a group of economic examples, concerning the concept of the ownership of a productive resource in a multiperson interactive context. Recently, Martin Shubik (a Yale economist and RAND consultant) and I investigated about a dozen such models, reflecting different institutional forms of ownership. All started with the same technology--a simple production function--yet when analyzed as n-person games their characteristic functions proved to be quite different.

There are two inputs, say land and labor, and one output, say corn. We shall first assume that one man owns all the land, so that the output is just a function $f(x)$ of how many laborers work on the field (Chart 4).

n + 1 PLAYERS:

"0" — owns the land

1, 2, ..., n — contribute labor

CHARACTERISTIC FUNCTION:

$$v(S) = f(s-1) \quad \text{if "0" in } S$$

$$v(S) = 0 \quad \text{if "0" not in } S$$

(S any coalition with s members)

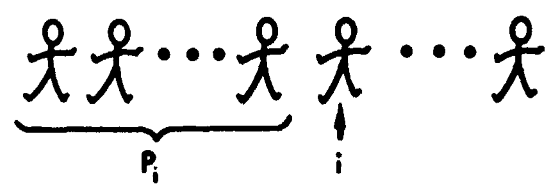
Chart 5—Model 1: The landless peasants

The characteristic function is easily found (Chart 5). A coalition not containing the landlord can produce nothing, while a coalition that contains the landlord can produce $f(s-1)$, where s is the size of the coalition. (The landlord does not work.)

To illustrate how we might "solve" this characteristic function, let us compute the value solution. The value to any player is defined as the average

amount he contributes to the worth of a coalition. A simple way to arrive at this average is to imagine the players shuffled into a random order, like a deck

RANDOM COALITION-BUILDING:



VALUE TO PLAYER i:

$$\phi_i = E\left[v(P_i \cup i) - v(P_i) \right]$$

(His average marginal worth)

Chart 6—Calculating the value

of cards, and then to build up a coalition one player at a time, until all are in (Chart 6). Let each player be paid the amount his inclusion adds to the worth of the growing coalition. What, then, are the <u>expected payoffs</u>, taking into

account the random shuffle? Despite the artificiality of this scheme, the expected payoff to each player comes out to be equal to the value of the game to him.

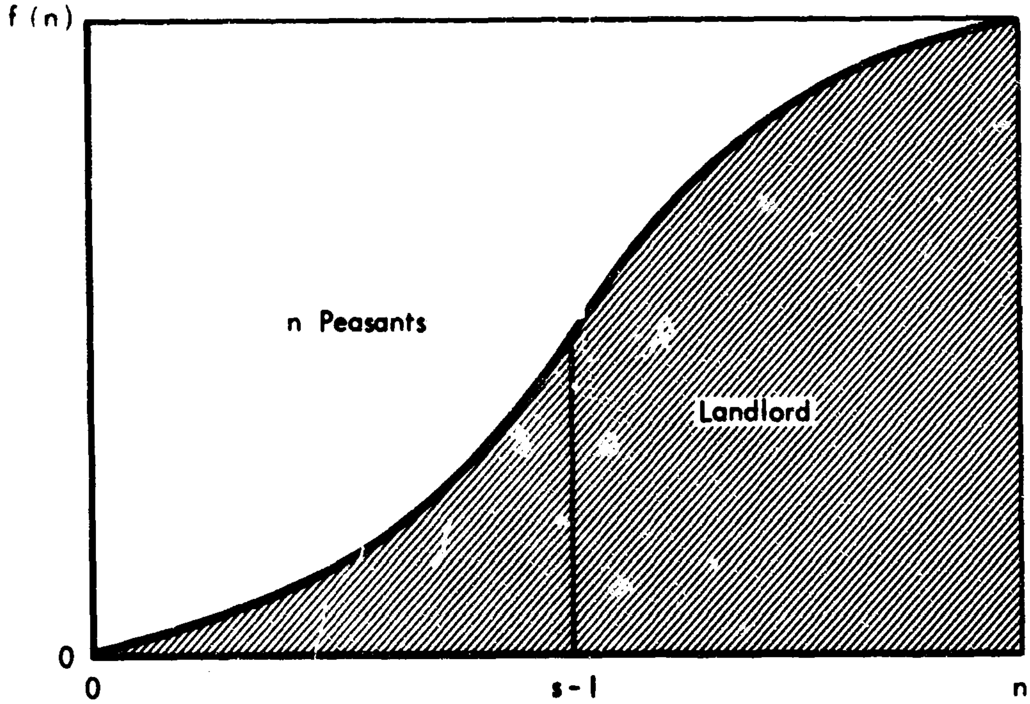


Chart 7—Model 1: Apportionment of the value

In the present example, the values are easy to come by. Start with the landlord. In the random shuffle, he will turn up in any position, from 1 to n+1, with equal probability. At position s, there will be s-1 peasants preceding him, and his entry into the coalition will boost production from zero to f(s-1), the height of the curve at that point. Averaging over s, we see that his value is essentially just the area <u>under</u> the production curve. Here the whole rectangle represents the total value of the game. The peasants' values, therefore, divide the remaining area (Chart 7).

Observe how this solution depends on the entire production curve. This

is in marked contrast to the classical competitive equilibrium. In that approach,

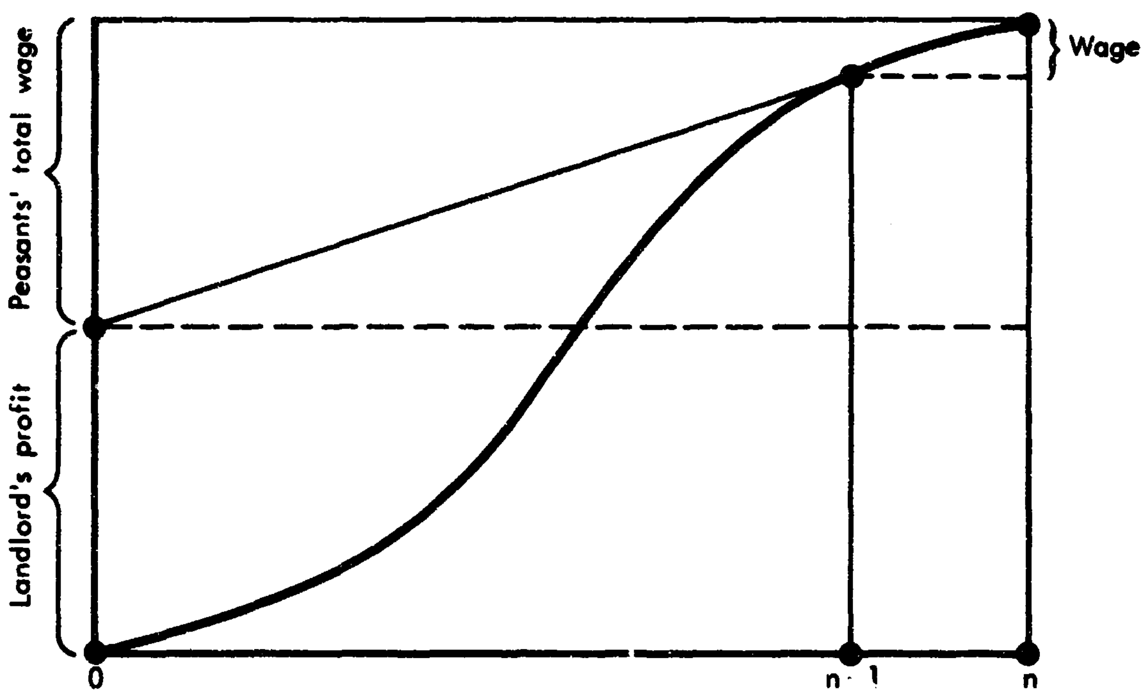


Chart 8—Model 1: The competitive solution

labor is regarded as a commodity for sale, and the price (wage) is equated to marginal productivity, i.e., the added amount of product due to the n-th laborer. The total wage bill is found by projecting a tangent line back to the y-axis, as shown. Because marginal productivity is less than average productivity in our example, the owner scores a profit, which may be either greater or less than his game value. For example, a small area under the curve means that he can

be "blackmailed" by moderate-sized coalitions of peasants, and his low value would reflect this vulnerable bargaining position. The classical solution, however, is sensitive only to the slope of the curve in the vicinity of the full-production point.

- PLOT SIZES: $c_1 + c_2 + \cdots + c_n = C$
- LABOR INPUTS: $\ell_1 + \ell_2 + \cdots + \ell_n = L$
- PRODUCTION FUNCTION $y = F(c, \ell)$

$$v(S) = F\left(\sum_S c_i, \sum_S \ell_i\right)$$

(Assuming increasing returns to scale)

Chart 9—Model 2: Individual ownership

In the second model, we abolish the landlord and give the land to the peasants. Since the symmetric case is not so interesting, we assume that the plots are different in size, and for further variety, we let the "players" (e.g., households) have different quantities of labor to contribute as well. Production is now expressed as a function of two variables: $F(c,\ell)$.

We may further assume that there are "economies of scale" implicit in the function F, so that whenever a coalition forms, they do best by pooling both their land and their labor. The characteristic function may then be written as follows:

$$v(S) = F\left(\sum_S c_i, \;\; \sum_S \ell_i\right)$$

This is an example of a class of games that has been studied extensively by mathematicians, using the methods of measure theory. In such a "measure game," the worth of a coalition depends only on the measures of one or more poolable resources brought in by the members. The value solution of a measure game has been shown to be proportional (approximately) to some weighted average of the measures--which are in this case the distributions of land and labor.

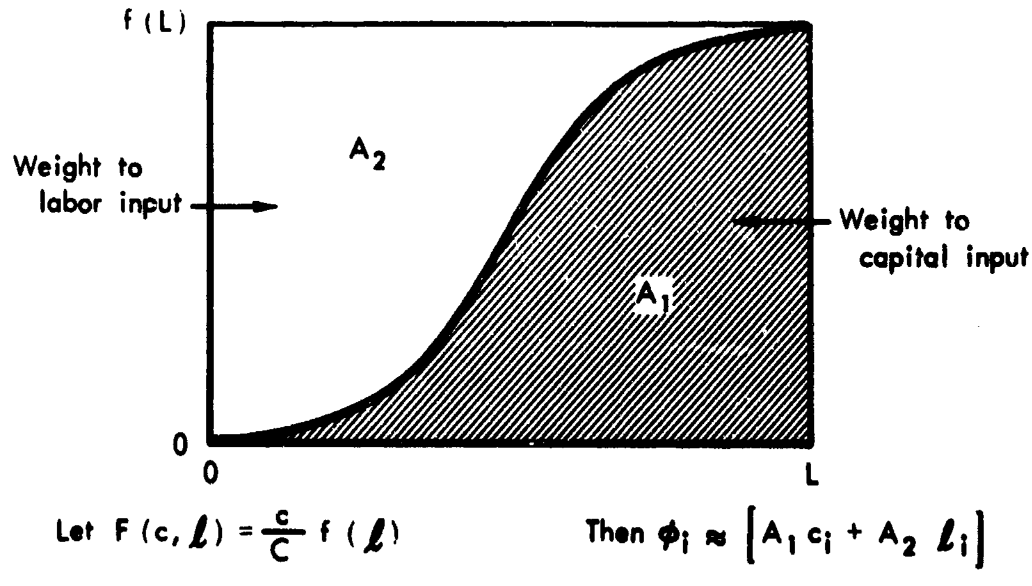


Chart 10—Model 2

The weights that arise in the present example can be shown quite simply if we assume that $F(c,\ell)$ is of the form $\frac{c}{C}f(\ell)$, with f as in the previous example (and $L = n$). The weight given the land input is then the area A_1, lying below the production curve, and the weight given the labor input is the area A_2, lying above it and to the left.

This is an intuitively satisfactory result. For example, if A_1 is large, then labor is unlikely to be in short supply, even if a subcoalition goes into business for itself, and a man's value should depend chiefly on the land he provides. But if A_1 is small, then labor is likely to be critical, and the value solution gives little weight to the distribution of landholdings.

Many other ownership forms can similarly be modeled as games (Chart 11). The modeling of corporate ownership, in particular, presents many delicate distinctions of strategic control, corporation objectives, rights of minority shareholders, etc. There is no time for details, but one suggestive result that Shubik and I found may interest you. In the model it is necessary to specify whether or not the corporation, in hiring workers, is allowed to discriminate in favor of members of the controlling coalition. The effect on the characteristic function is small, but significant; we found repeatedly that permitting discrimination wipes out the core of the game. That means, if you recall the definition, that the nondiscriminatory versions can be played in such a way that every coalition gets at least its characteristic value; but the discriminatory games cannot be so stabilized. Some disaffected coalition can always be found that has the power to improve its lot.

These models that I have discussed are admittedly oversimple and unrealistic. Game theory can of course be much more sophisticated. Yet I feel that its role at the present time does not lie in the direction of bigger and more elaborate models, which strive to deliver specific advice to specific decisionmakers faced with specific problems. Like any other scientific theory, its overriding objective must be to add to our understanding of the subject phenomena. For the

- FEUDAL SYSTEM
- VILLAGE COMMUNE: KIBBUTZ
- CORPORATE OWNERSHIP

  (SEVERAL VARIANTS OF
  DIVIDEND, WAGE, AND
  HIRING POLICIES)

Chart 11—Other ownership forms

time being, at least, n-person game theory seems destined to serve most effectively as a critic--destructively as often as constructively--in exposing the unspoken assumptions and opening up the blind spots of the more traditional "one-person" or purely behavioristic approaches to multilateral decisionmaking.

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Two references may be of interest: Ownership and the Production Function, RM-4053-1-PR, by L. Shapley and M. Shubik, and The "Value of the Game" as a Tool in Theoretical Economics, P-3658, by L. Shapley.